Induction

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Simple example: Suppose you see a queue, and

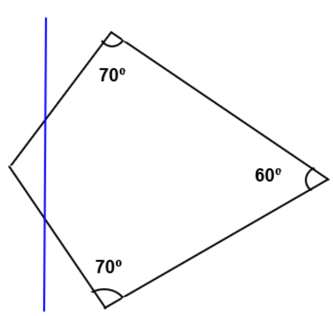
1) the first person in the queue is woman;

2) behind each woman there is a woman.

Is it true that there are only women in the queue?

Problem 1. Prove that for every $n \geq 3$ there exists a convex polygon that has exactly three acute angles.

Solution: For $n = 3$ the statement is obvious. For $n = 4$ it is easy to build an example with angles $\measuredangle 70°$,$\measuredangle 60°$,$\measuredangle 70°$ and $\measuredangle 160°$. Let $n = 5$, we will build a pentalygon with exactly 3 acute angles taking our example for $n = 4$ and cutting off its obstute angle so that we get a pentagon.

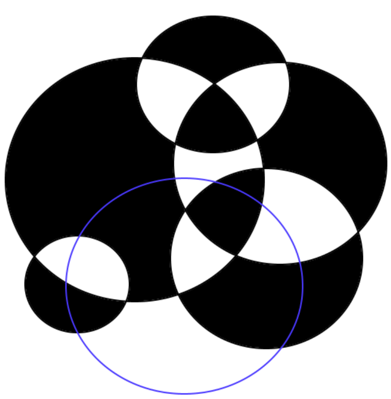


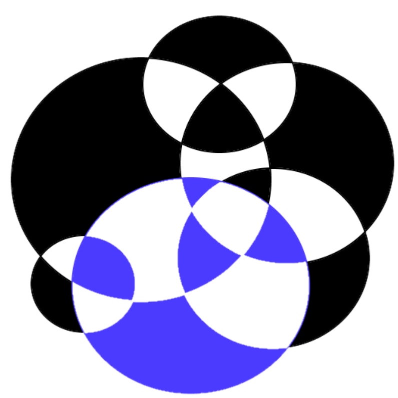
Clearly, the new shape has exactly three acute angles.

Problem 2. There are n circles in a plane. Prove that the regions in the plane divided o by the can be colored with two colors (black and white) in such a way that no two regions sharing some length of border are the same color

Solution: Let's start with one circle and two regions: one black and one white. Then let's add the second circle. It is easy to check that no matter where you draw the second circle you can still colour the regions in two colours.

Now draw the third circle and do not change any colors for now. There are three types of regions: inside the new circle, outside the new cirsce and croosed by the new circle. No two adjacent regions outside or inside of the new circle are painted in same color, because you did not change any colors, and any two regions on the inside or outside may share only a part of a circle which they shared before you drew a new circle. Moreover, if you invert the colors of all regions inside the new circle, then this property will still hold for them.

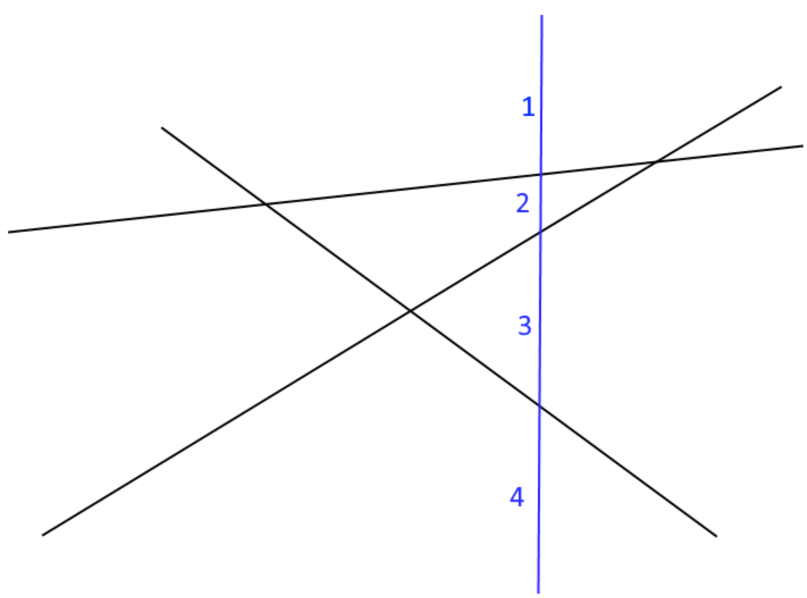




When you draw the third circle, you created new regions by dividing some regions in two parts. These and only these newly created adjacent regions will share same color. But once you invert the colors of all regions inside, you get what you need. All adjacent regions on either side are still painted in different colors, and now newly created adjacent regions (which share the same side which is a part of the new circle) are opposite colors as well, because one of them got inverted.

We can repeat the same consideration for the forth circle, then fifth and so on.

Problem 3. What is the maximum number of regions defined by n lines in the plane?



Solution: Let L_n denotes the maximum number of regions defined by n lines in the plane. Obviously $L_0 = 1$ and $L_1 = 2$. Short experimentation showes that $L_2 = 4$. Let's find $L_3 = 7$. If we want to obtain the maximum possible number of regions, we shall not let the new line pass through the intersection of the first two, for then we would get six regions and can do better. Leaving that point on one side of the third line means that the line won't be able to cross all four already existent regions, but at most only three - one more than there are lines. This gives a clue to a general case.

Suppose we have drawn n lines. When adding a line, the latter may cross each existent line in one point and thus cross at most $n+1$ regions. So $L_{n+1} = L_n + (n+1)$.

Let's find the formula for L_n:

$$L_n = n + L_{n-1} = n + (n-1) + L_{n-2} = n + (n-1) + (n-2) + \cdots + 2 + 1 + L_0$$

$$= \frac{n^2+n}{2} + 1 = \frac{n^2+n+2}{2}$$

Problem 4. Show that any number greater than 7 can be presented in the form $3x + 5y$, where x and y are positive.

Solution: Observe that

$8 = 3 + 5;$

$9 = 3 \times 3;$

$10 = 2 \times 5;$

For 11, 12 and 13 we will use the above identities and get:

$11 = 8 + 3;$

$12 = 9 + 3;$

$13 = 10 + 3.$

Any bigger number can be obtained by adding 3 several times to 8, 9 or 10.

As you have noticed in all the above problems we followed the following algorithm: first, we proved the statement for small numbers, say for $n = 1$, then we showed how to prove for $n = 2$, then for $n = 3$ and so on. This type of proof has special name: induction. Here is the formal way to prove statements by induction

Formal proof by induction:

The simplest and most common form of mathematical induction infers that a statement involving a natural number n holds for all values of n. The proof consists of two steps:

1. Base case: prove that the statement holds for the a small natural number n (usually $n = 1$).

2. The inductive step: prove that, if the statement holds for some natural number n, then the statement holds for $n + 1$.

The hypothesis in the inductive step that the statement holds for some n is called the induction hypothesis (or inductive hypothesis). To perform the inductive step, one assumes the induction hypothesis and then uses this assumption to prove the statement for $n + 1$.

Induction can be very useful for proving inequalities and identities.

Problem 5. Prove by induction $1 + 3 + 5 + \cdots + 2n - 1 = n^2$.

Solution: Let $a_n = 1 + 3 + 5 + \cdots + 2n - 1$.

Base case: $a_1 = 1 = 1^2$, so the statement holds for $n = 1$.

Inductive step: Suppose $a_n = n^2$. Consider

$$a_{n+1} = 1 + 3 + 5 + \cdots + 2n - 1 + 2(n+1) - 1 = a_n + 2n + 1$$

$$= n^2 + 2n + 1 = (n+1)^2.$$

So, by induction, we have proved the formula for all n.

Problem 6. Prove that $\sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6}}}} < 3$. Show it for arbitrary number of square roots.

Solution: Let $a_n = \sqrt{6 + \sqrt{6 + \dots \sqrt{6 + \sqrt{6}}}}$ (n square roots).

Base case: Let's show that $a_1 < 3$:

$$a_1 = \sqrt{6} < \sqrt{9} = 3.$$

Induction step: Suppose $a_n < 3$. Consider

$$a_{n+1} = \sqrt{6 + a_n} < \sqrt{9} = 3.$$

Exercises.

(1) Find 10 different numbers, sum of which is divisible by each of these numbers. (Hint: first find three such numbers).

(2) Square cannot be cut into 2 or 3 smaller squares, but it can be cut in 6 or 7 squares. Find all n such that square can be cut into n smaller squares.

(3) There are 100 houses in a village. How many fences can be built so that:

   (i) No two fences intersect;

(ii) Each fence surrounds at least on house;

(iii) There are no two fences that surround the same set of houses?

(4) The Tower of Hanoi is a mathematical game or puzzle. It consists of three rods, and a number of disks of different sizes which can slide onto any rod. The puzzle starts with the disks in a neat stack in ascending order of size on one rod, the smallest at the top, thus making a conical shape.

The objective of the puzzle is to move the entire stack to another rod, obeying the following simple rules:

(i) Only one disk can be moved at a time.

(ii) Each move consists of taking the upper disk from one of the stacks and placing it on top of another stack i.e. a disk can only be moved if it is the uppermost disk on a stack.

(iii) No disk may be placed on top of a smaller disk.

Find the minimum number of moves required to solve a Tower of Hanoi puzzle.

(5) A circle and a chord of that circle are drawn in a plane. Then a second circle, and chord of that circle, are added. Repeating this process, once there are n circles with chords drawn, prove that the regions in the plane divided o by the circles and chords can be colored with three colors in such a way that no two regions sharing some length of border are the same color.

(6) $2n$ dots are placed around the outside of the circle. n of them are colored red and the remaining n are colored blue. Going around the circle clockwise, you keep a count of how many red and blue dots you have passed. If at all times the number of red dots you have passed is at least the number of blue dots, you consider it a successful trip around the circle. Prove that no matter how the dots are colored red and blue, it is possible to have a successful trip around the circle if you start at the correct point.

(7) Show that the number $11\dots1$ (81 figures $'1'$) is divisible by 81.

(8) Prove by induction the formula for geometric progression

$$1 + x^2 + x^3 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}.$$

(9) Prove by induction

$$\frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n} < \frac{1}{\sqrt{n+1}}.$$

(10) (**) A sphere is covered with some number of caps which are hemispheres. Prove that it is possible to choose four hemispheres, and remove all others, while still keeping the sphere covered. (Hint: See Helly's theorem wikipedia)